A.) Mean Field Theory

i.) Quasilinear Theory

a.) Review of 1D/Basics

Refs:

① "Nonlinear Plasma Theory", by A. A. Galeev and R. Z. Sagdeev
↔ Reviews of Plasma Physics, Vol. 7 (Leontovich)
→ much better than book by same authors with same title

② "Plasma Turbulence", by B. B. Kadomtsev (1966)

③ "Cooperative Effects in Plasmas", by B. B. Kadomtsev
↔ Reviews of Plasma Physics, Vol. 22

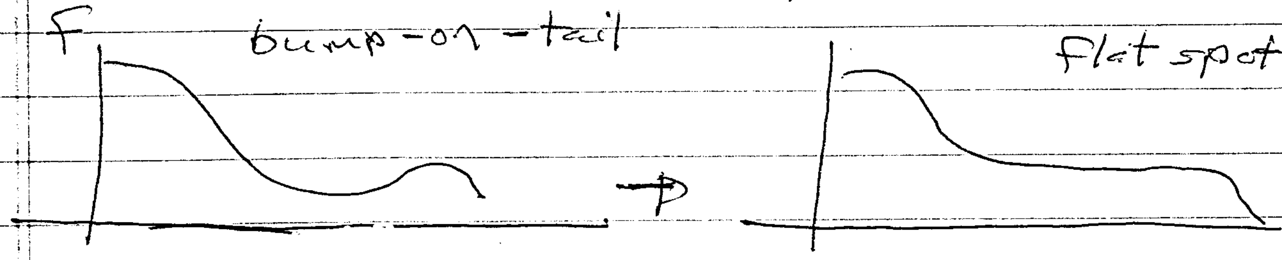
④ "Regular and Chaotic Dynamics", A. J. Lichtenberg and M. A. Lieberman; Springer-Verlag (1991).

⑤ "Hamiltonian Chaos and Fractional Dynamics" G. M. Zaslavsky, Oxford (2005)

①

⇒ Seek: How does $\langle f \rangle$ evolve due turbulence instability ?

i.e.



aim ⇒ nonlinear saturation
⇒ transport (v, x, ...) ⇒ turbulent transport coefficients
⇒ relaxation, structure formation

⇒ tend toward uniform, most probable, etc. state ?

if $\omega, k \gg 1/T, 1/L$

turbulence scales — mean scales

then can employ 2-scale strategy

i.e. $$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

⇒

$$\underbrace{\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial x}}_{\text{linear propagation}} + \underbrace{\frac{q}{m}\tilde{E}\frac{\partial \tilde{f}}{\partial v}}_{\text{nonlinearity}} = \underbrace{-\frac{q}{m}\tilde{E}\frac{\partial \langle f \rangle}{\partial v}}_{\text{drive from mean}}$$

linear propagation

nonlinearity
- mode-mode coupling
- trapping

drive from mean

$\Rightarrow$ fluctuations

and

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{q}{m}\frac{\partial}{\partial v}\langle \tilde{E}\tilde{f} \rangle$$

slow evolution

beats

$\Rightarrow$ mean

- reasonable, but closure problem?
  What to do about $\tilde{f}$

- QLT: ① assume all spectral constituents are eigenmodes
  i.e. all $\omega = \omega(k)$

  ② $\tilde{f}$ treated as linear response

i.e. plug:

$$\tilde{f}_k = \frac{-i\frac{q}{m}\tilde{E}_k\,\partial\langle f \rangle/\partial v}{\omega - kv} \rightarrow \text{linear response}$$

into $\langle f \rangle$ equation, yielding:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = \mathrm{re}\, \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{i}{\omega - kv}$$

$$= \frac{q^2}{m^2} \sum_k |E_k|^2 \left\{ \frac{|\gamma_k|}{(\omega_k - kv)^2 + \gamma_k^2} \right\}$$

Note:

→ $|\gamma_k|$ → causality (i.e. D from spectrum of damped modes)

→ resonant, non-resonant

$$\frac{|\gamma_k|}{(\omega_k - kv)^2 + \gamma_k^2} \to \pi \delta(\omega_k - kv) + \frac{|\gamma_k|}{\omega_k^2}$$

→ can formulate full set of "quasilinear equations" → describes evolution via mean relaxation i.) $\langle f \rangle \Rightarrow$ linear stability, via

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial \langle f \rangle / \partial v}{\omega - kv} \Rightarrow \begin{matrix}\omega_{\underline{k}} \\ \gamma_{\underline{k}}\end{matrix}$$

ii.) spectral evolution

$$\frac{\partial}{\partial t}|E_{\underline{k}}|^2 = 2\gamma_{\underline{k}}|E_{\underline{k}}|^2 \Rightarrow |E_{\underline{k}}|^2$$

iii.) $\langle f \rangle$ modification / evolution

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = \mathrm{Re} \sum_{\underline{k}} \frac{q^2}{m^2}|E_{\underline{k}}|^2 \frac{i}{\omega - kv} \quad = \langle f \rangle$$

$\Rightarrow$ convergence to marginal profile (?, $\rightarrow$ plateau?).

$\rightarrow$ Amazingly QLT works (as description of transport) quite well ....

{though not universally

WHY?

② ⇒ Numbers and Ratios ......

- what is assumed in QLT?

→ linear response adequate – no NL distortion

→ resonant diffusion – Markov process, a la' FPE
→ stochasticity / irreversibility
→ RPA (?)

Exercise:

a.) Derive QL (resonant diffusion) equation from Fokker-Planck theory

b.) use Hamiltonian structure of dynamics to eliminate dynamical friction term (c.f. Lichtenberg and Lieberman)

⇒ $\underline{\underline{2}}$ dimensionless numbers:

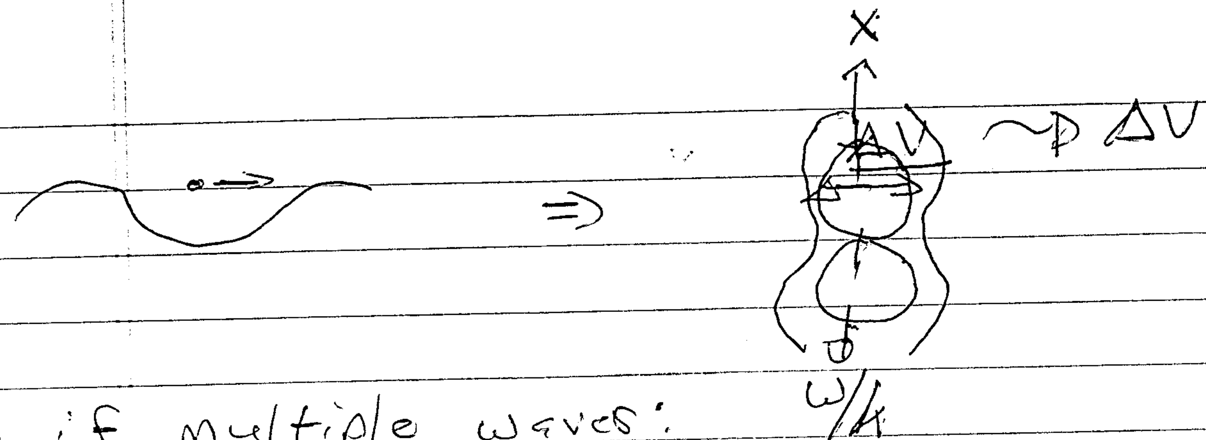
①

$S_c = \Delta V / |v_{ph\,i} - v_{ph\,i+1}|$ → measures stochasticity of particle orbits

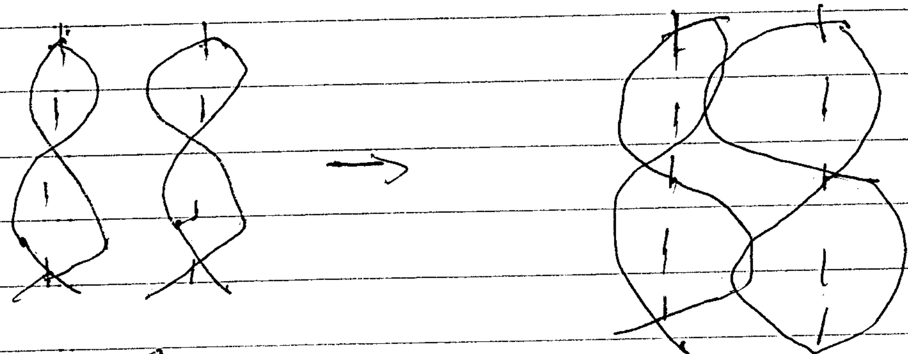
↓

Chirikov #

i.e. resonances overlap



so if multiple waves:



$\delta < 1$ $\qquad\qquad\qquad\qquad$ $\delta > 1$

⇒ ~ independent resonant behavior

⇒ resonance overlap

⇒ diffusive "kicking" from 1 resonance to another.

(2) Kubo #, Strouhal # → measures memory of flow

→ field correlation time

i.e.

$$K = \frac{(qE/m)\,\tau_c}{(\Delta V)_c}$$

→ velocity correlation length

for spatial scattering:

$$S = \frac{\tilde{V}\,\tau_c}{l_c}$$

$\tau_c \to$ scatterer correlation time

$l_c \to$ correlation length

Note: For $\Delta V_c \sim \Delta V_T \sim \sqrt{2E/m}$

$$K \sim k(\Delta V_T)\,\tau_c$$

$$\sim \omega_b\,\tau_c$$

$\omega_b \to$ bounce freq.

$\Rightarrow$

$K, S > 1 \Rightarrow$ { ordered flow with memory
persistent scatterer pattern

$K, S < 1 \Rightarrow$ { random, short lived scatterers
random flow — suggests RPA

usually $K, S > 1 \Rightarrow$ linear unperturbed trajectory
<u>poor</u> approximation

$\Rightarrow$ need incorporate scattering field.

so, usual wisdom is that QLT valid if:

→ $\sigma_c > 1$ → need stochastic orbits

→ $K < 1$ → need avoid trapping, strong distortion, etc.

but: ?

① → is field/scatter correlation time the relevant time scale ?

② → what of $\sigma_c > 1$, $K \sim 1$?

③ → what of non-resonant piece ?

Regarding ①,

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty e^{i(\omega - kv)\tau} d\tau$$

$$|E_k|^2 = E_0^2 \, \Delta k \Big/ \left[(k - k_0)^2 + (\Delta k)^2 \right]$$

⇒

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty d\tau\, e^{i(\omega - kv)\tau}$$

$$= \int dk \frac{q^2}{m^2} \frac{E_0^2 \Delta k}{(k-k_0)^2 + (\Delta k)^2} \int_0^\infty d\tau\, e^{i(\omega - kv)\tau}$$

$$\approx \frac{q^2}{m^2} E_0^2 (2\pi) \int_0^\infty d\tau\, e^{i(\omega_{k_0} - k_0 v)\tau} \times$$

$$\exp\left[-\left| \frac{d\omega}{dk} - v \right| \Delta k\, \tau \right]$$

sets correlation decay

$\Rightarrow$ if $D \sim \frac{q^2}{m^2} \langle E^2 \rangle \tau_c$

wave-particle correlation time

then:

$1/\tau_{ac} \sim \left| \frac{d\omega}{dk} - v \right| \Delta k \rightarrow$ wave-particle decorrelation rate

if $v \sim \omega / k$ – resonance

$1/\tau_{ac_{pkt}} \sim \left| \frac{d\omega}{dk} - \frac{\omega}{k} \right| \Delta k \rightarrow$ packet dispersal rate

Generally: $\tau_{ac_{w-p}} \neq \tau_{ac_{pkt}} \rightarrow$ differences more pronounced in 3D

So, really need:

→ specify velocity for wave-particle decorrelation being considered.

→ need: $1/\tau_{ac} > \omega_b,\ 1/\tilde{\tau}_{scatt}$

$$\tilde{\tau}_{scatt}^{-1} \sim (k^2 D)^{1/3}$$

→ $\tau_{ac} \sim \tilde{\tau}_c$ only for resonant particles in 1D.

→ broad spectrum alone is not sufficient to justify QLT → effective dispersion significant!

$$1/\tau_{ac}\Big|_{res} \sim \left|\frac{d\omega}{dk} - \frac{\omega}{k}\right| \Delta k \rightarrow 0, \text{ for non-dispersive waves}$$

→ $K<1$ criterion most accurate if one takes $\tilde{\tau}_c \sim \tau_{ac,pkt}$

$$K = \frac{q}{m} E\, \tau_{ac,pkt}/\Delta v_t = (k\Delta v_t)\tau_{ac} < 1$$

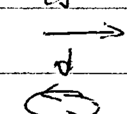
agrees with intuition.

→ "phase randomization" irrelevant ⇒

- can have $\sigma > 1$, $K < 1$ with coherent phases

- QL known to well describe stochastic trajectory divergence in standard map/ magnetic field lines, even for <u>static</u> fields/ fixed phases. cf: Rechester, Rosenbluth, White PRL '80.

- phases fixed in Tsunoda/Malmberg experiments

→ often, QLT seems to work reasonably well in limit of $\sigma > 1$, $K \sim 1$
  - unclear why....
  - corrections due granulations needed (?!)
    <u>i.e.</u> $K \sim 1 \Rightarrow$  $\Rightarrow$ 
    phase space eddys formed....

→ strong non-stationarity can (boost) applicability of QLT.

Exercise :

→ consider, starting from $\underline{B} \cdot \underline{\nabla} \psi = 0$, a heuristic equation for the density of magnetic field lines in a "cylindromak":

$$B_T \frac{\partial N}{\partial z} + \frac{B_\theta(r)}{r} \frac{\partial N}{\partial \theta} + \tilde{B} \cdot \nabla N = 0$$

$$\frac{\partial N}{\partial z} + \frac{B_\theta(r)}{r B_T} \frac{\partial N}{\partial \theta} + \frac{\tilde{B}}{B} \cdot \nabla N = 0$$

from $$\frac{dr}{\tilde{B}_r} \doteq \frac{r\,d\theta}{B_\theta + \tilde{B}_\theta} = \frac{dz}{B_T}$$

→ derive the QL equation for radial diffusion of $\langle N \rangle$

→ what is the QL diffusion coefficient

→ derive expressions for K, S

→ what is the analogue of wave-particle resonance here ...

③ Energetics

- easy to show (c.f. supplementary notes)

RPKED → resonant particle kinetic energy density

$$= \mathcal{E}_R = \int_{res} dV \, \frac{1}{2} m v^2 f$$

WED → wave energy density

$$= \int dk \, \omega_k N_k \qquad N_k = \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$\frac{\partial}{\partial t}\left(\mathcal{E}_R + \int dk \, \omega_k N_k \right) = 0$$

- since WED = FED + NRPKED

FED → field energy density $\sim |E_k|^2/8\pi$

NRPKED → non-resonant particle kinetic energy density

then, equivalently can write:

$$\frac{\partial}{\partial t}\left(\mathcal{E}_p + \mathcal{E}_F \right) = 0$$

Exercise: Prove these.

- why important?
  - basic consistency requirements
  - physical picture $\Rightarrow$ i.e. "quasi-particle picture"

i.e. represent plasma as:

+ { - resonant particles $\Rightarrow$ { Vlasov eqn. + $\delta(\omega - kv)$
  - waves $\rightarrow$ quasiparticles $\Rightarrow$ WKE }

$$\frac{\partial N}{\partial t} + (\underline{v}_{gr} + \underline{V}) \cdot \nabla N - \frac{\partial}{\partial x}(\omega + \underline{k} \cdot \underline{V}) \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

$\Rightarrow$ natural, multi-component picture

$\Rightarrow$ useful for accounting for { momentum / energy } transport by waves.

i.e. $\underline{P}_w = \int d\underline{k}\, \underline{k}\, N$

$\mathcal{E}_w = \int dk\, \omega_k N$

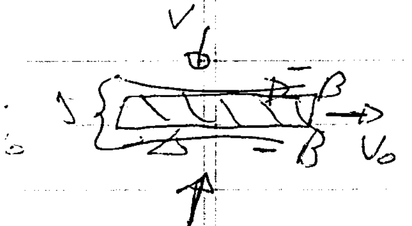
$\rightarrow$ { can compute wave-induced stresses, etc.
  formulate radiation hydro, etc. }

Anomalous Resistivity → $\left\{\begin{array}{l}\text{Application} \\ \text{of} \\ \text{QLI}\end{array}\right.$ $\left\{\begin{array}{l}\text{Follows / expands} \\ \text{on Galeev / Sagdeev} \\ \text{Rev. Pl. Phys. 67.}\end{array}\right.$

→ an instructive and important example of quasilinear theory is anomalous resistivity

→ here, try approach classic current-driven ion acoustic instability (CDIA) model of anomalous resistivity via coupled micro-macro dynamics

– consider Sweet-Parker model, i.e.



$\langle E \rangle = \langle \frac{v B}{c} \rangle < 0$

Cirito Pegel

$$\left.\begin{array}{l} VL = V_{out}\Delta \\ \\ V_{out} = V_A \\ \\ 2V\frac{B^2}{8\pi}L = \eta \bar{J}^2 L\Delta \end{array}\right\} \text{cf } 218B \text{ notes}$$

$$\frac{\Delta}{L} = \frac{V}{V_A} \Rightarrow \boxed{\Delta^2 = \frac{(L)(\eta)}{V_A}} \Rightarrow \text{layer width}$$

$$\frac{\Delta}{L} \sim 1/\sqrt{R_m}$$

What happens as η decreased ?

$$-\frac{c}{4\pi\Delta}B = J = n q \bar{V}_e$$

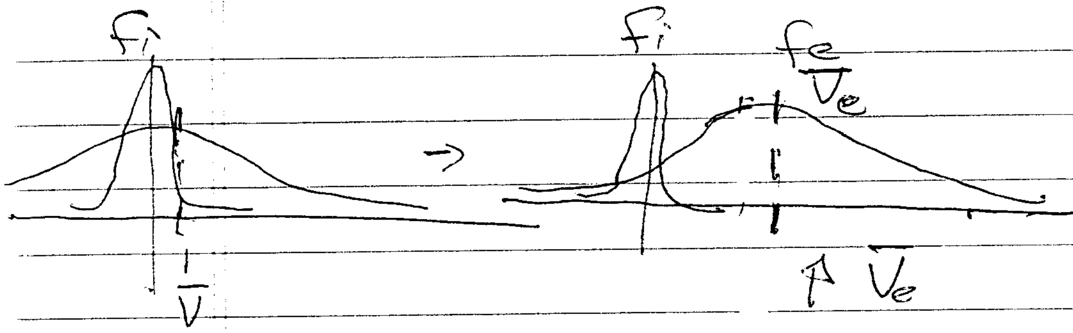
$\bar{V}_e$: electron drift speed

$$\text{so} \quad \bar{V}_e = cB/4\pi n q \Delta = \frac{d_{skin,e}}{\Delta}\Omega_e$$

Now, $\bar{V}_e \sim B/\Delta n \Rightarrow \bar{V}_e \uparrow$ as

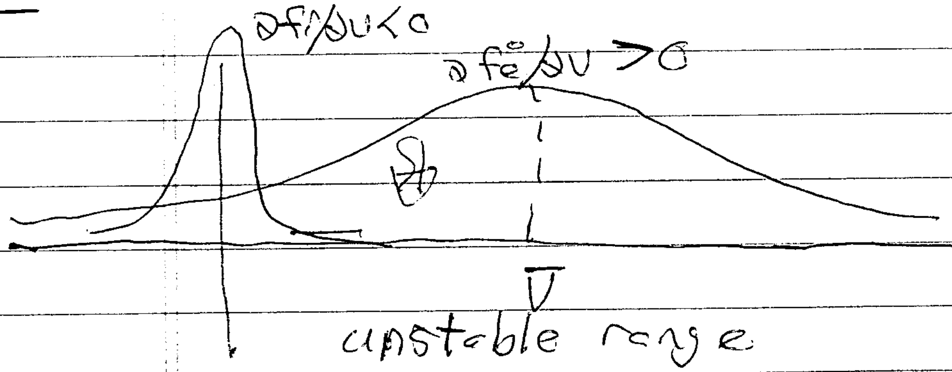
- $\Delta \downarrow \Rightarrow$ narrow layer
- $n \downarrow \rightarrow$ few charge carriers
- $B \uparrow \Rightarrow$ stronger field (drive)

$\Delta \uparrow \Rightarrow$



$\Rightarrow$ decreasing Δ raises $\bar{V}$
$\rightarrow$ up-shifts f_e centroid relative to f_i
$\Rightarrow$ destabilizes CDIA !

<u>i.e.</u> classic scenario of CDIA



$\therefore$ expect CDIA will:

$\rightarrow$ exchange momentum between electrons and waves

<u>so</u> $\Rightarrow$ slow down electrons, reduce $\bar{V}_e$

$\rightarrow$ act as "anomalous", "turbulent" resistivity i.e. $\begin{cases} \Delta^2 = \dfrac{L}{V_A}\left(\eta + \eta_A(\overline{V})\right) \\ \quad\quad \hookrightarrow \text{anomalous resistivity} \\ \overline{V} = \dfrac{cB}{4\pi} / nq\Delta \end{cases}$

How calculate:

(a) Brute Force

- confining oneself to 1D model, ignoring Layer structure, have:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{\varepsilon}{m} E \frac{\partial f}{\partial V} = C(f)$$

here $x \to$ vertical, $v \to$ vertical velocity

vertical $\to$ $\perp$ to layer

$m_e V * \Rightarrow$

$$\frac{\partial}{\partial t}\langle P_e\rangle - q\left\langle E \int dv\, f\right\rangle = -\nu_{e,i} n_0 m_e \overline{V}_e$$

collisional loss to ions

$$\frac{\partial \langle P_e\rangle}{\partial t} - q n_0 \langle E\rangle - q\langle \tilde{E}\tilde{n}\rangle = -\nu_{e,i} n_0 m_e \overline{V}_e$$

so

$$\langle E\rangle + \left\langle \tilde{E}\frac{\tilde{n}}{n_0}\right\rangle - \frac{1}{n_0 q}\frac{\partial \langle P_e\rangle}{\partial t} = +\frac{\nu_{ei} m_e}{n_0 q^2} n_0 q \overline{V}_e$$

at ⓒ stationary state,

$$\langle E\rangle + \langle \tilde{E}\frac{\tilde{n}}{n_0}\rangle = \eta\langle J\rangle$$

driving field $\sim\langle VB\rangle$

electron acceleration by turbulence

⇒ "anomalous resistivity"

collisional resistivity

to calculate:

$$\langle \tilde{E}\frac{\tilde{n}}{n_0}\rangle = \sum_{k} +ik\phi_{-k}\frac{\tilde{n}_k^e}{n_0}$$

$$= \int dv \sum_{k} ik\phi_{-k}\,\tilde{f}_k^e$$

electron density perturbation

$f_k^e \to f_k^{eL}$

- quasilinear calculation
- stationarity ⇒ resonant transport.

ⓑ Conservation Argument

- as in ⓐ anticipate stationarity ⇒ resonant quasilinear evolution

- recall,

$$\frac{\partial}{\partial t}\left(\mathcal{E}^{RP} + \mathcal{E}^{wave}\right) = 0$$

$$\frac{\partial}{\partial t}\left(P^{RP} + P^{wave}\right) = 0$$

$$\mathcal{E}^{w}_{\underline{k}} = \omega_{\underline{k}} \left.\frac{\partial\epsilon}{\partial\omega}\right|_{\underline{k}} \frac{|E_{\underline{k}}|^2}{8\pi} \equiv \omega_{\underline{k}} N_{\underline{k}}$$ → # quanta

$$P^{w}_{\underline{k}} = \frac{k}{\omega} \mathcal{E}^{w}_{\underline{k}} = k N_{\underline{k}}$$

as waves (CDIA) electrostatic, can ignore field momentum.

so, for resonant electrons

$$\frac{\partial}{\partial t} P_e^{RP} = -\frac{\partial P^w}{\partial t} = -\sum_{\underline{k}} (2\gamma^e_{\underline{k}}) \frac{k}{\omega_{\underline{k}}} \mathcal{E}^w_{\underline{k}}$$

$\gamma^e_{\underline{k}} \equiv$ electron (resonant) growth rate

but

$\frac{\partial}{\partial t} P^{RP}_{electron}$ ⇒ slowing down

⇒ macro-representation as effective collision frequency

so $\underbrace{\frac{\partial P^{RP}_{electron}}{\partial t}}_{\text{slowing down by resonant scattering (resonant particle interaction)}} = -n m_e \underbrace{\nu_{eff}}_{\text{effective collision frequency}} \bar{V}$

$$\boxed{n m_e \nu_{eff} \bar{V} = \sum_k (2\gamma^e_k) \frac{k}{\omega_k} \mathcal{E}^{\omega}_k}$$ *

- defines $\bar{V}$
- for macro-micro link

$$\bar{V} = \frac{cB}{4\pi n q \Delta}$$

* - n.b. if 2D, 3D theory, i.e. ⊥ dynamics ⇒ non-resonant scattering ⇔ wave driven momentum flux i.e. $\Pi_{\perp,\parallel}$ → ⊥ radiation ∥ momentum. Relation to whistler interpretation of Bellan?! There need include wave radiation in energy balance.

so now have

$$\boxed{\begin{aligned} n\, m\, \nu_{eff}(B, \Delta)\bar{V} &= \sum_k (2\gamma_k^e)\frac{k}{\omega_k}\,\mathcal{E}_k^w \\ \Delta^2 &= \frac{b}{V_A}\left(\eta + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right) \end{aligned}} \qquad (1)$$

$\Rightarrow$ need γ_k^e, $\mathcal{E}_k^w$ and $\langle f^e\rangle$ evolution

at simplest level, proceed via linear/quasilinear theory in 1D

- at more advanced level:
  - consider 1D phase space structures
    - $\to$ electron/ion clumps, momentum exchange
    - $\to$ electron scattering off ion hole
  - consider 3D $J_\parallel$ driven instability with electron viscosity

Now, proceed in usual fashion:

$\gamma_k^e \to$ linear theory

$\mathcal{E}_k^w \to$ nonlinear saturation

$\langle f_e\rangle \to$ QL equation — flattening

For linear theory of CDIA ;

$$\nabla^2 \hat{\phi} = -4\pi n_0 |e| \left(\frac{\hat{n}_i}{n_0} - \frac{\hat{n}_e}{n_0} \right)$$

$$\hat{n}_i / n_0 = \frac{k^2 c_s^2}{\omega^2} \frac{|e|\phi}{T}$$

$$\frac{\hat{n}_e}{n_0} = \frac{|e|\hat{\phi}}{T} \left[1 - i\Gamma(k) \right]$$

$\Gamma(k)$,

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} = -\frac{|e|}{m_e} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\hat{f} = \frac{|e|\hat{\phi}}{T} \langle f \rangle + g$$

$$\frac{\partial \tilde{g}}{\partial t} + v \frac{\partial \tilde{g}}{\partial x} = -v \frac{\partial}{\partial x} \left(\frac{|e|\hat{\phi}}{T} \langle f \rangle \right) + \frac{|e|}{m_e} \frac{\partial \tilde{\phi}}{\partial x} \frac{\partial \langle f \rangle}{\partial v}$$
$$- \frac{\partial}{\partial t} \left(\frac{|e|\hat{\phi}}{T_e} \langle f \rangle \right)$$

$$= v \frac{\partial \tilde{\phi}}{\partial x} \frac{|e|}{T} \langle f \rangle + \frac{|e|}{\cancel{m_e}} \frac{\partial \phi}{\partial x} \left(-\frac{(v - \bar{v})}{T/\cancel{m_e}} \right) \langle f \rangle - \frac{\partial}{\partial t} \frac{|e|\hat{\phi}}{T_e} \langle f \rangle$$

$$= -\frac{\partial}{\partial t} \frac{|e|\hat{\phi}}{T} \langle f \rangle + \bar{v} \frac{\partial}{\partial x} \frac{|e|\hat{\phi}}{T} \langle f \rangle$$

$$\Rightarrow g_k = \frac{i(\omega - k\bar{V})}{-i(\omega - kV)} \frac{|e|\hat{\phi}_k}{T} \langle f \rangle$$

$$= -\left(\frac{\omega - k\bar{V}}{\omega - kV}\right) \frac{|e|\hat{\phi}_k}{T} \langle f \rangle$$

$$\omega_k^{2(0)} = \frac{k^2 c_s^2}{1 + k^2 \lambda_D^2}$$

$$-i\,r(k) = \int dV + \frac{(\omega - k\bar{V})}{(\omega - kV)} \langle f \rangle$$

$$\bar{f} = \frac{1}{\sqrt{\pi}} \exp\left[\frac{-(\omega/k - \bar{V})^2}{V_{Th}^2}\right]$$

$$= -(\omega - k\bar{V}) \frac{(+i\pi)}{|k|V_{Th}} \bar{f}\Big|_{\omega/kV_{Th}}$$

$$1 + k^2\lambda_D^2 = \frac{k^2 c_s^2}{\omega^2} + \frac{(\omega - k\bar{V})}{|k|V_{Th}} (i\pi) \bar{f}\Big|_{\omega/kV_{Th}}$$

$$\omega \to \omega + \delta\omega$$

$$0 = -\frac{2\delta\omega}{\omega} + (\omega - k\bar{V}) \frac{(-i\pi)}{|k|V_{Th}} \bar{f}\Big|_{\omega/kV_{Th}}$$

$$\frac{\delta\omega}{\omega} = -\frac{i\pi}{2} \frac{(\omega - k\bar{V})}{|k|V_{Th}} \bar{f}\Big|_{\omega/kV_{Th}}$$

$\delta\omega \Rightarrow i\gamma_k$ growth rate

(2) $$\boxed{\gamma_k \sim \frac{-\pi}{2} \omega_k \frac{(\omega - k\bar{V})}{|k|V_{Th}} \bar{f}\Big|_{\omega/kV_{Th}}}$$ $\Rightarrow$ $\gamma > 0$ for $\bar{V} > c_s$

$\Rightarrow$ critical velocity

for $\langle f \rangle$ evolution,

$$\frac{\partial \langle f \rangle}{\partial t} = +\frac{\partial}{\partial v} \sum_k + \frac{|e|}{m_e} \tilde{E}_{-k} \tilde{g}_k$$

$$= +\frac{\partial}{\partial v} \sum_k \frac{|e|}{m_e} \tilde{E}_{-k} \left(\frac{-(\omega - k\bar{v})}{(\omega - kv)} \frac{|e|\phi_k}{T} \langle f \rangle \right)$$

$$= \frac{\partial}{\partial v} \sum_k \frac{T_e |e|}{m_e} \frac{+ik\phi_k}{T_e} \left(-(\omega - kv) i\pi\delta(\omega - kv) \right) \frac{|e|\phi_k}{T} \langle f \rangle$$

$$= \frac{\partial}{\partial v} \sum_k (-v_{th}^2) \left| \frac{|e|\phi_k}{T} \right|^2 k(\omega - k\bar{v}) \, \pi\delta(\omega - kv) \langle f \rangle$$

$$\boxed{\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} \sum_k (-v_{th}^2) \left| \frac{|e|\phi_k}{T} \right|^2 k(\omega - k\bar{v}) \, \pi\delta(\omega - kv) \langle f \rangle} \quad (3)$$

– mean evolution

Note:

– really only assumed $\langle f \rangle = \langle f((v-\bar{v})^2/2v_{th}^2) \rangle$

so $$\frac{\partial \langle f \rangle}{\partial v} = \left(\frac{v - \bar{v}}{v_{th}^2} \right) \langle f \rangle$$

and

$$\langle f \rangle' = -\langle f \rangle$$

→ minimal assumption on structure

- can write as $\bar{v}$ evolution

$$\bar{v} = \int dv\, v \langle f \rangle \Big/ \int dv \langle f \rangle$$

$\Rightarrow$

$$\boxed{\frac{\partial \bar{v}}{\partial t} = + \int dv \sum_k k_{\mp h}^2 \left| \frac{e\phi_k^\omega}{T} \right|^2 k^2 \left(\frac{\omega}{k} - \bar{v} \right) \pi \delta(\omega - kv) \langle f \rangle}$$

$\omega/k < \bar{v} \rightarrow \partial \bar{v}/\partial t < 0$
$\quad\; > \bar{v} \rightarrow \partial \bar{v}/\partial t > 0$

Remains to determine fluctuation intensity level

Generically, can write:

$$\frac{\partial}{\partial t} \mathcal{E}_k^w = \underset{ⓐ}{\gamma_k \mathcal{E}_k^w} - \underset{ⓑ}{\left(\sum_{k'} \omega_k\, C_1(k, k') \frac{\mathcal{E}_{k'}^w}{nT} \right) \mathcal{E}_k^w}$$

$$- \underset{ⓒ}{\left(\sum_{k', k''} \omega_k\, C_2(k, k', k'') \frac{\mathcal{E}_{k'}^w}{nT} \frac{\mathcal{E}_{k''}^w}{nT} \right) \mathcal{E}_k^w}$$

Spectral equation constituents:

ⓐ – linear growth

ⓑ – quadratic nonlinearity $\rightarrow$ 3 wave coupling / NL ion-wave interaction

ⓒ – "cubic NL" $\rightarrow$ wave coupling

Now, for ion-acoustic wave:

- 3 wave coupling effects negligible
  ⇒ can't satisfy resonance
- NL wave-particle effects weak ⇒ intrinsically
⇒ consider 4 wave process

$$\boxed{\frac{\partial \mathcal{E}_{\underline{k}}^{\omega}}{\partial t} = \left[\gamma_{\underline{k}} - \omega_{\underline{k}}\, B(\omega, \underline{k}) \left(\frac{\mathcal{E}_{\underline{k}}^{\omega}}{nT} \right)^2 \right] \mathcal{E}_{\underline{k}}^{\omega}} \qquad (5)$$

- 'cartoon' NL saturation equation

Now, (4)-(5) ⇒ { coupled, (2)-stationary micro-macro system

⇒ describe anomalous resistivity dynamics and its effect on reconnection

⇒ Coupled solution corresponds to solution of the problem

$$\textcircled{1}\begin{cases} n m_e \nu_{eff}(B,\Delta)\,\bar{V} = \sum_k 2\gamma_k^e \frac{k}{\omega_k} \mathcal{E}_k^w \\ \Delta^2 = \frac{L}{V_A}\left(\eta + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right), \quad \bar{V} = cB/4\pi n q \Delta \end{cases}$$

$$\textcircled{2}\ \gamma_k^e = -\frac{\pi}{2}\,\omega_k\,\frac{(\omega - k\bar{V})}{|k|\,v_{th}}\,\bar{f}\Big|_{\omega/k v_{th}}$$

$$\textcircled{4}\ \frac{\partial \bar{V}}{\partial t} = \int dv\left(\sum_k v_{th}^2 \left|\frac{e\tilde{\phi}_k}{T}\right|^2 k^2 \left(\frac{\omega}{k} - \bar{V}\right) \pi\delta(\omega - kv)\langle f\rangle\right)$$

$$\textcircled{5}\ \frac{\partial \mathcal{E}_k^w}{\partial t} = \left[\gamma_k - \omega_k\,B(\omega,k)\left(\frac{\mathcal{E}_k^w}{nT}\right)^2\right]\mathcal{E}_k^w$$

Now, stationarity $\Rightarrow$

$$\mathcal{E}_k^w = nT\left(\gamma_k/\omega_k B\right)^{1/2}$$

$$\gamma_k = +\frac{\pi}{2}\left(\frac{\bar{V} - c_s}{|k\,v_{th}|}\right) k\,\omega_k\,\bar{f}\Big|_{\omega/k v_{th}}.$$

so, for scalings:

$$\nu_{eff} = \frac{1}{nm\bar{V}} \sum_k 2\gamma_k^e \frac{k}{\omega_k} \mathcal{E}_k^{w}$$

$$\sim \frac{1}{nm\bar{V}} \frac{(\bar{V}-C_s)}{|k|v_{Th}} k\cancel{\omega_k} \bar{f}\Big|_{\frac{\omega_k}{kv_{Th}}} \frac{k}{\cancel{\omega_k}}(nT)\left(\frac{\gamma_k}{\omega_k B}\right)$$

$$\sim \frac{(1-C_s/\bar{V})}{\cancel{n}m} \frac{k^2\cancel{n}T}{|k|v_{Th}}\left(\frac{\gamma_k}{\omega_k B}\right)\bar{f}\Big|_{\frac{\omega_k}{kv_{Th}}}$$

$$\sim (1-C_s/\bar{V})\,\bar{f}\Big|_{\omega/kv_{Th}}\left(\frac{k^2}{|k|}v_{Th}\right)\left(\frac{\gamma_k}{\omega_k B}\right)$$

$$\sim (1-C_s/\bar{V})\left(\frac{k\cancel{\omega_k}}{|k v_{Th}|}\frac{(\bar{V}-C_s)}{\cancel{\omega_k}B}\bar{f}\right)^{1/2}\bar{f}\,\frac{k^2 v_{Th}}{|k|}$$

$$\sim \left[(\bar{V}-C_s)k\right]^{3/2}\bar{f}^{3/2}\frac{k(v_{Th}/\bar{V})}{|k|\,|kv_{Th}|^{1/2}}$$

$$\boxed{\nu_{eff} \sim \frac{\left[(\bar{V}-C_s)k\right]^{3/2}}{|kv_{Th}|^{1/2}}\frac{v_{Th}}{\bar{V}}\bar{f}^{3/2}\left(\frac{k}{|k|}\right)}$$

Turbulent collision frequency

so have

$$\boxed{\begin{aligned} \Delta^2 &= \frac{L}{V_A}\left(\eta + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right) \\ \bar{V} &= cB_0 / 4\pi n_0 q \Delta \\ \nu_{eff} &= \frac{\left[(\bar{V}-C_s)k_\parallel\right]^{3/2}}{|k v_{Th}|^{1/2}} \frac{v_{Th}}{\bar{V}} \bar{F}^{3/2} \left(\frac{k_\parallel}{|k_\parallel|}\right) \end{aligned}}$$

with:

$$\boxed{\begin{aligned} \frac{\gamma_k}{\omega_k} &= \frac{\pi}{2}\left(\bar{V}-C_s\right)\frac{k_\parallel}{|k_\parallel| v_{Th}} \bar{F} \\ \bar{F} &= \frac{1}{\sqrt{\pi}} \exp\left[-\left(\omega/k_\parallel - \bar{V}\right)^2 / 2 v_{Th}^2\right] \end{aligned}}$$

→ characterize micro-macro coupling with anomalous resistivity

→ now can envision situation
- finite current, $\Delta \sim (L\eta/L_A)^{1/2}$
  $\nu_{eff} = 0$

so if:

- decrease $\eta \Rightarrow \Delta$ decreases
- Δ decreases $\Rightarrow \bar{V}$ increases

- $\bar{V}$ increases $\Rightarrow \gamma_k > 0$

- $\gamma_k > 0 \Rightarrow \begin{cases} \Sigma_k^{\omega} > 0 \\ \nu_{eff} > 0 \end{cases} \Rightarrow$

- $\nu_{eff} > 0 \Rightarrow \begin{cases} A \text{ increases} \\ \bar{V} \text{ decreases} \end{cases}$

$\Rightarrow$ decreasing ν so A decreases triggers feedback so A increases $\rightarrow$ selfregulation / $\oplus$ feedback

$\therefore$ in this model, can expect:

- at low $\nu_{collisional}$, so $\bar{V}\big|_c \sim cB_0/4\pi n_0 e \Delta_c \sim C_s$

$\Rightarrow$ CDIA "hovers" near marginal stability

- for stronger drive (i.e. $B_0 \uparrow$)

$\Rightarrow$ ion interaction important

$\Rightarrow$ strong ion distortion possibly significant

$\Rightarrow$ granulation formation important

$\Rightarrow$ distortion of electron distribution function need be considered

i.e.